Mentoring Advanced Placement
Mathematics

Topic: 3 dimensions and beyond

Challenge:

1.) The intersection of two circles can be a point, two points, or a circle. The intersection of two spheres can be a point, a circle, or a sphere.

- What does the intersection of two 4-D spheres look like? Three 4-D spheres?

2.) In a plane, squares can completely fill the 2-D space if placed side by side in all directions. Equilateral triangles, and regular hexagons can also accomplish this. Cubes and regular octahedrons act similarly in filling 3-D space.

- Can regular tetrahedrons fill in 3-D space without gaps?
- Dodecahedrons? Icosahedrons?

Challenge Solutions:

1.) A 4-D sphere intersecting our 3-D world can be imagined as a point that suddenly appears, becoming a larger and larger sphere. Then, it gets smaller, becomes a point and disappears. (This is similar to a 3-D sphere passing through a 2-D plane. It appears as a point, then an increasingly large circle, decreasing to a point, and then gone.)

Two 4-D spheres could grow in parallel, touching at a point. They could grow overlapping with a circle. If superimposed, they could grow while spherically overlapping at each stage. They could also be off set and appear as nested shells.

Three 4-D spheres would have mostly the same results, but produce a set of three nested shells if each sphere did not coincide exactly with the others.

This quandary has an analog in theoretical physics. A proposed model of quarks holds that they exist in 4-D space, and that protons are composed of three quarks that coincide and intersect spherically.

2.) In three dimensions, tessellation makes a jungle-gym framework of polyhedrons expanding to infinity in all directions. If all polyhedrons must be regular and identical, the only regular matrix will come from cubes.

If combinations of two different regular polyhedrons are allowed, the space can be filled either by regular tetrahedrons or octahedrons. At each vertex, eight tetrahedrons will fit flush, and six octahedrons will do the same. Interestingly, these patterns of stacking resemble the basic pattern for many natural crystals.